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# Walking solitons in type II second-harmonic generation

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We find the families of spatial walking solitons propagating in quadratic nonlinear media under conditions for type II second-harmonic generation in the presence of Poynting vector walk-off between the interacting beams. The analytical stability criterion for these three-parameter vector solitons is established. We study the shape and general properties of the solitons and their stability on propagation. It is found that the stationary solutions are *stable* at moderate positive phase mismatch. At phase matching and negative phase mismatch there are some unstable solutions near the cutoff. [S1063-651X(97)51612-6]

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Parametric optical solitons in quadratic nonlinear media [1] have attracted growing attention, because of their potential applications to the manipulation of light signals in different scenarios [2–9]. Such solitons exist in a wide variety of material and input light conditions in both bulk crystals and planar waveguides, and recently spatial solitons consisting of two strongly coupled optical fields have been observed experimentally in second-harmonic generation (SHG) [10,11]. Another recent contribution to this rapidly developing field is the demonstration of the spatial modulational instability of wide elliptical beams and multisoliton generation in quadratic nonlinear media [12]. In general, except under suitable conditions, in the low-power regime the parametrically interacting beams propagate along different directions due to the Poynting vector walk-off present in anisotropic media. However, when a soliton is formed the beams mutually trap and in the presence of Poynting vector walk-off they propagate locked together. Such a beam locking opens the possibility to specific applications of the solitons [13,14], and it also poses new questions to the understanding of the soliton formation. The ''walking'' solitons existing in the presence of walk-off exhibit new features in comparison to the nonwalking solitons, and investigation of these features is important from both applied and fundamental viewpoints. This is so because the approach has implications to walking solitons existing in other physical settings, such as birefringent optical fibers [15]. To date, two-parameter families of spatial walking solitons in quadratically nonlinear media have been studied under conditions for type I phase matching, in both (1+1)-dimensional [16] and (2+1)-dimensional geometries [17]. Two-parameter families of (1+1)-dimensional temporal chirped solitons in nonlinear quadratic media have been also investigated by using a somewhat different approach [18].

In this Rapid Communication we investigate the formation of spatial walking solitons under conditions for type II SHG; therefore, the solitons exist due to the mutual trapping of an ordinary polarized fundamental frequency (FF) beam, an extraordinary polarized FF beam, and the extraordinary polarized second harmonic (SH) beam. Such solitons constitute a three-parameter family and they exist for different total energy, for different unbalancing energy between the two fundamental beams, and for different soliton velocities. We study the amplitude and phase-front shapes of the solitons, derive the cutoff conditions for the families of solutions, give the stability criterion for the three-parameter family of solitons, and verify numerically its predictions.

The normalized evolution equations for the slowly varying field envelopes in type II phase-matching geometries for second-harmonic generation can be written as

$$
\begin{aligned}
&i\frac{\partial a_1}{\partial \xi}-\frac{\alpha_1}{2}\frac{\partial^2 a_1}{\partial s^2}+a_3a_2^*\exp(-i\beta\xi)=0,\\
&i\frac{\partial a_2}{\partial \xi}-\frac{\alpha_2}{2}\frac{\partial^2 a_2}{\partial s^2}-i\delta_2\frac{\partial a_2}{\partial s}+a_3a_1^*\exp(-i\beta\xi)=0,\\
&i\frac{\partial a_3}{\partial \xi}-\frac{\alpha_3}{2}\frac{\partial^2 a_3}{\partial s^2}-i\delta_3\frac{\partial a_3}{\partial s}+a_1a_2\exp(i\beta\xi)=0,
\end{aligned}
\tag{1}
$$

where $a_1$, $a_2$, and $a_3$ are the normalized envelopes of the ordinary polarized FF beam, the extraordinary polarized FF beam, and the SH beam, respectively. For spatial solitons and for the relevant experimental conditions (a characteristic beam width $\eta\sim 15$ $\mu$m, a coherence length $l_c\sim 2.5$ mm, a diffraction length $l_d\sim 1$ mm, and a walk-off angle $\rho\sim 1°$) one obtains $\alpha_1=-1$, $\alpha_2=k_1/k_2\simeq -1$, and $\alpha_3=k_1/k_3\simeq -0.5$, the mismatch parameter $\beta\sim\pm 3$, the walk-off parameters $\delta_{2,3}\sim\pm 1$ and $\xi\simeq 20$ correspond to a few centimeters. Equations (1) have several conserved quantities, namely, the total energy flow

$$
I=I_1+I_2+I_3=\int\left(\frac{1}{2}|A_1|^2+\frac{1}{2}|A_2|^2+|A_3|^2\right)ds, \tag{2}
$$

the energy unbalancing $I_b=I_1-I_2$, the transverse beam momentum

$$
\begin{aligned}
J=J_1+J_2+J_3=\frac{1}{4i}\int\Bigg[&\left(A_1^*\frac{\partial A_1}{\partial s}-A_1\frac{\partial A_1^*}{\partial s}\right)+\left(A_2^*\frac{\partial A_2}{\partial s}\right.\\
&\left.-A_2\frac{\partial A_2^*}{\partial s}\right)+\left(A_3^*\frac{\partial A_3}{\partial s}-A_3\frac{\partial A_3^*}{\partial s}\right)\Bigg]ds,
\end{aligned}
\tag{3}
$$

and the Hamiltonian

 

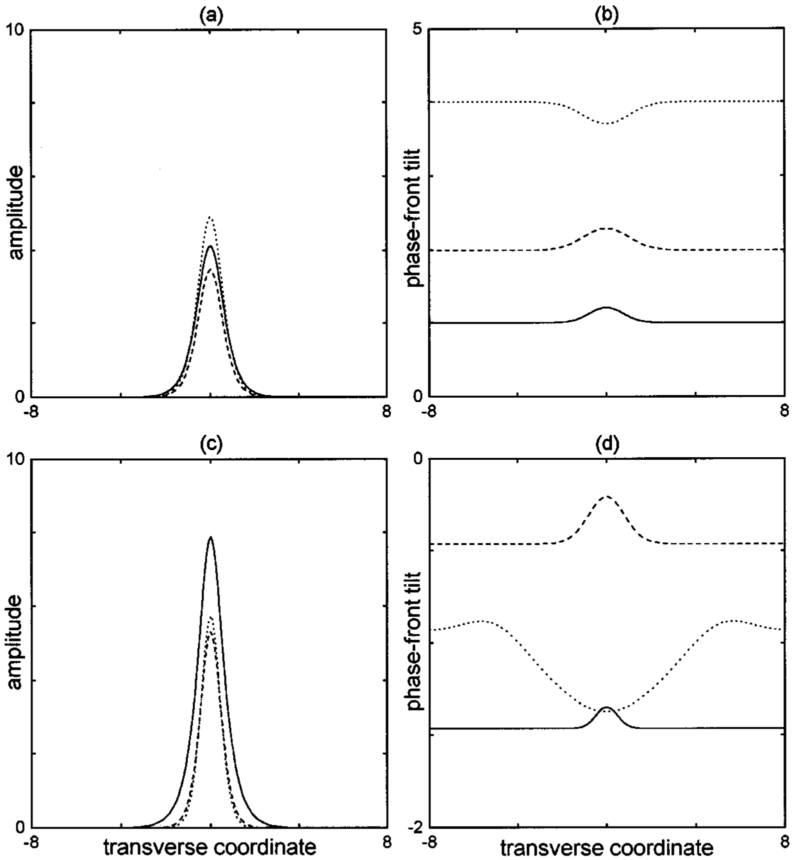


FIG. 1. Amplitude and local phase-front tilt of two solitons walking with different velocities at negative wave vector mismatch, versus transverse coordinate. In both cases $\beta=-3$, $\kappa_1=3$, and $\kappa_1/\kappa_2=0.5$. In (a) and (b) $v=1$. In (c) and (d) $v=2(1-\sqrt{3})$. Solid lines: ordinary polarized FF waves; dashed lines: extraordinary polarized FF waves; dotted lines: extraordinary polarized SH waves.

$$H=-\frac{1}{2}\int\left[\frac{\alpha_1}{2}\left|\frac{\partial A_1}{\partial s}\right|^2+\frac{\alpha_2}{2}\left|\frac{\partial A_2}{\partial s}\right|^2+\frac{\alpha_3}{2}\left|\frac{\partial A_3}{\partial s}\right|^2-\beta|A_3|^2\right.$$
$$+(A_1^*A_2^*A_3+A_1A_2A_3^*)+\frac{i}{2}\delta_2\left(A_2\frac{\partial A_2^*}{\partial s}-A_2^*\frac{\partial A_2}{\partial s}\right)$$
$$\left.+\frac{i}{2}\delta_3\left(A_3\frac{\partial A_3^*}{\partial s}-A_3^*\frac{\partial A_3}{\partial s}\right)\right]ds, \qquad (4)$$

where we have defined $A_1=a_1$, $A_2=a_2$, and $A_3=a_3\exp(-i\beta\xi)$.

We study stationary solutions of Eqs. (1) of the form $a_\nu(\xi,s)=U_\nu(\eta)\exp[i\phi_\nu(\xi,\eta)]$, $\nu=1,2,3$, with $U$ and $\phi$ being real functions, $\eta=s-v\xi$ is the transverse coordinate moving with the soliton peak, and $\phi_\nu(\xi,s)=\kappa_\nu\xi+f_\nu(\eta)$. Here $v$ is the soliton velocity, $\kappa_\nu$ the nonlinear wave-number shifts produced by the wave interaction, and $f_\nu(\eta)$ stand for the transverse phase-fronts of the solitons. A simple analysis of the above equations shows that in order to avoid all power exchange between the waves one first needs $\kappa_3=\kappa_1+\kappa_2+\beta$. Also the phase-fronts should verify either $f_3(\eta)=f_1(\eta)+f_2(\eta)$ everywhere or alternatively $U_\nu(\eta)$ and $f_\nu(\eta)$ have to be symmetric and antisymmetric functions of $\eta$, respectively. Only the trivial traveling-wave solution that occurs in the absence of the walk-off fulfills the former condition, whereas the walking solitons fulfill the latter. The trivial traveling-wave solution, which has no physical relevance unless $\alpha_3=-0.5$ and there is no walk-off, has flat phase-fronts $f_\nu(\eta)=\omega_\nu\eta$, where $\omega_1=-v/\alpha_1$, $\omega_2=-(\delta_2+v)/\alpha_2$, $\omega_3=-(\delta_3+v)/\alpha_3$. The physically relevant family of walking solitons has nontrivial, curved phase-fronts. The local phase-front tilts are defined as $df_\nu/d\eta$, $\nu=1,2,3$. Their values at infinity are as follows: $-v/\alpha_1$, $-(\delta_2+v)/\alpha_2$ and $-(\delta_3+v)/\alpha_3$. Families of type II SHG walking solitons parametrized by the nonlinear wave-number shifts $\kappa_1$ and $\kappa_2$, and soliton velocity $v$ exist at different values of material parameters $\alpha_{2,3}$, $\beta$, and $\delta_{2,3}$, with different energy flows, and in a domain of parameter space $(\kappa_1,\kappa_2,v)$ such that $\kappa_1\geqslant-v^2/(2\alpha_1)$, $\kappa_2\geqslant-(v+\delta_2)^2/(2\alpha_2)$, and $\kappa_1+\kappa_2\geqslant-\beta-(v+\delta_3)^2/(2\alpha_3)$. We will make numerical calculations on ''slices'' of the above three-dimensional parameter domain. Thus, calculating on the lines $\kappa_1=\rho\kappa_2$, and varying $\kappa_1$, $\rho$, and $v$ we cover the entire three-dimensional parameter domain. In the absence of walk-off, zero-velocity type II SHG parametric solitons were recently investigated [7,8]. The shape of the walking solitons depends strongly on the various parameters involved, and in particular on their velocity. For numerical calculations we set $\alpha_1=\alpha_2=-1$, $\alpha_3=-0.5$, and $\delta_2=\delta_3=1$. In Fig. 1 we have plotted the amplitude and the local phase-front tilt for two representative solitons in the case of negative phase mismatch. For the parameters chosen in Figs. 1(c)–1(d), although the mismatch is negative the amplitude of the ordinary polarized FF beam is larger than the amplitude of the SH beam. Notice that the two solitons plotted in Fig. 1 correspond to the same wave-number shifts $\kappa_{1,2}$, but that they carry significantly different total energies.

Stationary, walking soliton solutions of Eqs. (1) exist for nonlinear wave-number shifts $\kappa_1$, $\kappa_2$ and transverse velocities $v$ such that the soliton is not in resonance with linear

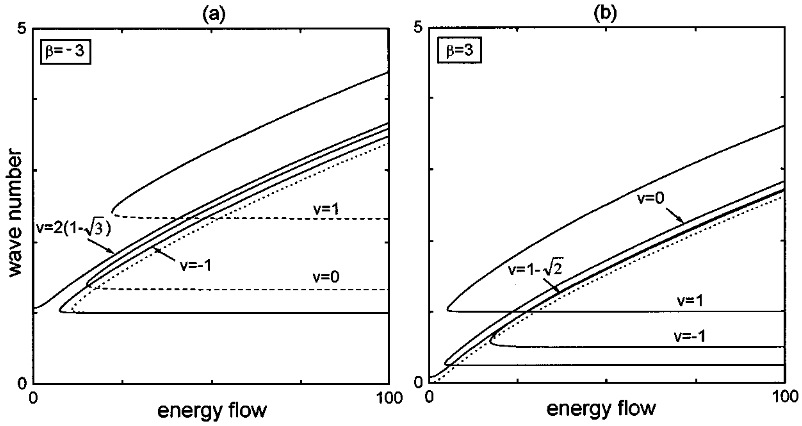


FIG. 2. Nonlinear wave number $\kappa_1$ as a function of the energy flow for the families of walking solitons for different soliton velocities. In all cases $\kappa_2=2\kappa_1$. (a) The solitons at negative phase mismatch ($\beta=-3$); (b) the solitons at positive phase mismatch ($\beta=3$). Dashed lines: unstable solutions. Dotted lines: nonwalking solitons ($\delta_2=\delta_3=0$, and $v=0$).

dispersive waves. Otherwise, the coupling between the waves would lead to energy leakage that would appear as Cherenkov radiation emitted from the soliton [19]. When the ratio of the two wave numbers $\kappa_1$, $\kappa_2$ is fixed, the resonance condition can be calculated to obtain

$$\kappa_{1,\mathrm{cut}}=\max\left\{\frac{1}{2}v^2,\rho\frac{(\delta_2+v)^2}{2},\frac{\rho}{1+\rho}[-\beta+(v+\delta_3)^2]\right\}, \quad (5)$$

where $\rho=\kappa_1/\kappa_2$. For given values of the various involved parameters, walking solitons exist for nonlinear wave-number shifts above this cutoff. When the maximum in Eq. (5) is attained on one of the three quantities involved, we have found that there is an energy threshold for the formation of the walking solitons. When the maximum is attained on at least two of these quantities, there is no energy threshold. Figure 2 shows the wave number $\kappa_1$ as a function of the total energy flow $I$ for two representative values of the wave vector mismatch $\beta$ and for the ratio of the wave numbers $\rho=\kappa_1/\kappa_2=0.5$. For other values of the parameter $\rho$, the results are similar to those shown in Fig. 2. We have also included in Fig. 2 the curves corresponding to the nonwalking type II SHG solitons [7,8]. The corresponding plot at exact phase matching ($\beta=0$) looks similar to those shown in Fig. 2, thus it is not presented here. With a few exceptions, at both negative and positive phase mismatches, the type II SHG walking solitons exist only for energy flows exceeding certain threshold values, somehow in contrast to some of the type I SHG walking solitons at positive phase mismatch for which there is no energy threshold for their formation [16]. However, it is worth emphasizing that this result only holds for the particular case shown here, but not for the families of solitons at the other possible values of the various involved parameters. For our choice of the parameters here, there is a pair of velocities for which there is no threshold for the soliton formation: $v=2(1\pm\sqrt{3})$ for $\beta=-3$, and $v=1\pm\sqrt{2}$ for $\beta=3$. This result is a direct consequence of the cut-off formula (5). Notice that in practice the absence of energy threshold for a particular family of solitons has a limited relevance, because low-energy solitons have a correspondingly large width, so that for an experimentally reasonable input beam width there is always a threshold for the excitation of solitons.

Important information about the families of walking solitons can be obtained from the conserved quantities as follows. The dynamical equations (1) constitute an infinite-dimensional Hamiltonian system and can be written in the canonical form $i\partial A_\nu/\partial\xi=2\,\delta_F H/\delta_F A_\nu^*$, $\nu=1,2,3$, where $\delta_F$ stands for the Frèchet derivative. The stationary walking solitons occur at the extrema of the Hamiltonian for given energy flows $I_u=I_1+I_3/2$, $I_v=I_2+I_3/2$, and a given transverse momentum $J$, i.e., they occur at $\delta_F\{H+\kappa_1 I_u+\kappa_2 I_v-vJ\}_{\mathrm{sta}}=0$. Next using either Derrick's theorem or directly manipulating the governing equations, one finds that that actually the stationary walking soliton solutions are realized at the value of the Hamiltonian $H=-(3/5)(\kappa_1 I_u+\kappa_2 I_v)+(1/5)\beta I_3+(4/5)vJ-(1/5)(\delta_2 J_2+\delta_3 J_3)$. We see that in the presence of the walk-off only the fourth term vanishes for zero-velocity solitons, whereas the last two terms contribute to the Hamiltonian. This is an indication that the transverse momentum of the walking solitons is not simply proportional to their velocity, opposite to walking solitons of Galilean invariant evolution equations. The actual relation between the velocity and the momentum can be found by examining the evolution of the energy centroid of the coupled state constituted by the two FF beams and the SH beam. Thus the velocity of type II SHG walking solitons is given by $v=[J-\delta_2 I_2-\delta_3 I_3-(\alpha_2+1)J_2-(2\alpha_3+1)J_3]/I$.

Next we derive the condition of marginal stability of the family of solutions by applying a multiscale asymptotic method [7]. Let $\mathbf{c}_0=\mathbf{c}_0(\eta)=(u_r^s,v_r^s,w_r^s,u_i^s,v_i^s,w_i^s)^T$ be the column vector formed with the stationary walking solitons

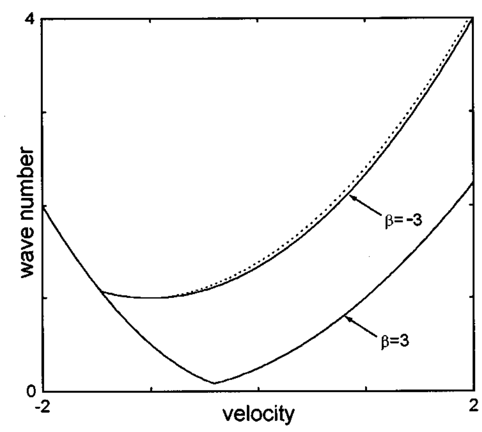


FIG. 3. Domains of existence and stability of the families of walking solitons in the plane $(\kappa_1,v)$. Here $\kappa_2=2\kappa_1$ and $\delta_2=\delta_3=1$.

$u_0=u_r^s+iu_i^s$, $v_0=v_r^s+iv_i^s$ and $w_0=w_r^s+iw_i^s$, corresponding to the two FF beams and the SH beam, respectively. To analyze the stability of these solutions with respect to small perturbations, we substitute $\mathbf{c}(\eta,\zeta)=\mathbf{c}_0(\eta)+\varepsilon\mathbf{c}_1(\eta)e^{\lambda\zeta}$, where $\mathbf{c}_1=(u_{1r},v_{1r},w_{1r},u_{1i},v_{1i},w_{1i})^T$ and $\varepsilon$ is a small parameter, into Eqs. (1) and linearize the resulting equations obtaining a linear eigenvalue problem $L\mathbf{c}_1=\lambda\mathbf{g}$, where $L$ is a self-adjoint operator and $\mathbf{g}=(u_{1i},v_{1i},w_{1i},-u_{1r},-v_{1r},-w_{1r})^T$. For $\lambda=0$ this eigenvalue problem has three spatially localized solutions $\partial\mathbf{c}_0/\partial\eta$, $(-u_i^s,0,-w_i^s,u_r^s,0,w_r^s)^T$, and $(0,-v_i^s,-w_i^s,0,v_r^s,w_r^s)^T$ giving the neutrally stable modes of the linear eigenvalue problem. In order to find a threshold condition for the linear instability we consider that the instability growth rate $\lambda$ is small, so that we can seek solutions of the above linear eigenvalue problem in the form of asymptotic series in the small parameter $\lambda$: $\mathbf{c}_1=\Sigma_{j=0}^{\infty}\lambda^j\mathbf{c}_1^{(j)}$, where $\mathbf{c}_1^{(j)}=(u_r^{(j)},v_r^{(j)},w_r^{(j)},u_i^{(j)},v_i^{(j)},w_i^{(j)})^T$. We introduce also the series $\mathbf{g}=\Sigma_{j=0}^{\infty}\lambda^j\mathbf{g}^{(j)}$, where $\mathbf{g}^{(j)}=(u_i^{(j)},v_i^{(j)},w_i^{(j)},-u_r^{(j)},-v_r^{(j)},-w_r^{(j)})^T$. Substituting the above expansions into the linearized equations and collecting terms of the same order in $\lambda$, we find the following explicit analytical solutions for the first-order corrections $\partial\mathbf{c}_0/\partial\kappa_1$, $\partial\mathbf{c}_0/\partial\kappa_2$, and $\partial\mathbf{c}_0/\partial v$. The instability threshold condition emerges at the next, second-order in $\lambda$. Thus, in the second order we get $L\mathbf{c}_1^{(2)}=\mathbf{g}^{(1)}$. Next we use the following property of a self-adjoint operator $L$: let $\mathbf{a}_0$ belong to the kernel space of the operator $L$ ($L\mathbf{a}_0=0$) and let $\mathbf{b}$ belong to the image space of the operator $L$ ($L\mathbf{a}=\mathbf{b}$), then the vectors $\mathbf{a}_0$ and $\mathbf{b}$ are orthogonal to each other. By imposing these orthogonality conditions we are left with a linear homogeneous system of equations, and its solvability condition gives the equation defining the threshold between stable and unstable three-parameter solitary waves,

$$\frac{\partial(I_u,I_v,J)}{\partial(\kappa_1,\kappa_2,v)}=0, \tag{6}$$

where $\partial(E,F,G)/\partial(x,y,z)$ stands for the Jacobian of $E,F,G$ with respect to $x,y,z$. Taking into account terms of the next order in $\lambda$ in the above asymptotic series will lead to equations describing linear and nonlinear regimes of the dynamics of unstable solitons, but this is beyond the scope of this paper. The stability criterion (6) can also be derived using geometrical approaches [20,21].

The output of stability analysis is summarized in Fig. 3 for two representative values of the wave vector mismatch and for a fixed ratio $\rho$ of the wave numbers. For positive mismatch and in the range of velocities shown in Fig. 3, all the solitons are stable above their existence line (the lower solid curve in Fig. 3). For negative mismatch, all the solitons that exist above the instability threshold (the dotted line in Fig. 3) are stable, whereas the very narrow region between the upper solid line (the existence or cutoff line) and the dotted line corresponds to unstable solitons. At exact phase matching, as for negative mismatch, there is also a very narrow region near the cutoff condition corresponding to unstable solitons. For other values of the parameter $\rho$, one obtains results similar to those shown in Fig. 3. To monitor the evolution of the stable and unstable solutions, we performed the corresponding series of numerical experiments by propagating a variety of stationary solutions. The numerics showed that the unstable solitons either spread or they reshape and excite a stable walking soliton. To further confirm the robustness of type II SHG walking solitons, we have verified that the solitons with different velocities form under a variety of excitation conditions, e.g., with tilted input beams, consistent with the results of previous numerical and experimental related investigations [14].

In conclusion, we have described the three-parameter families of (1+1)-dimensional spatial solitary waves due to three-wave mixing in a quadratic nonlinear medium under conditions for type II SHG, in the presence of Poynting vector walk-off between the interacting waves. The type II SHG walking solitons have curved phase-fronts, with nontrivial phase-front curvatures and exist for different soliton velocities and power flows. The remarkable properties of these solitons might open new possibilities and applications of solitons in quadratic nonlinear media and of walking solitons in analogous but different physical settings.

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